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Generalised Bloch Equations of Some Dynamical Quantum Systems

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Abstract

Generalised model operator Bloch equations are derived for some models in the field of quantum optics. These include:- (i) The behaviour of a single 2-level atom coupled with a broadband squeezed vacuum (phase-sensitive) reservoir outside the familiar "rotating wave approximation". (ii) The bistable behaviour in a ring cavity configuration subject to an extra "intense" coherent field as part of its environmental reservoir. In case (i) the average atomic inversion exhibits "transient and steady oscillatory" behaviour while in case(ii) a further "non-linear structure" shows in the characteristic input-output state relation due to field-dependent relaxation processes.

I- Introduction

Simplicity and idealisation in mathematical modelling of physical systems are essential as a basic step to study the behaviour of such systems. Improvement towards realistic modelling is usually stimulated by experimental results and scientific data. The field of quantum optics (quantum operator treatment of matter and radiation interaction), since the invention of the laser in 1960's, is rich of continuous experimental achievements which stimulate the improvement of mathematical modelling (e.g. [1, 2]). For our purpose, we quote two scientific experimental achievements, namely, (i) The femtosecond time resolution technique [3] to probe the very initial stage ($\approx 10^{-15}$ s.) of the spontaneous radiation by an atomic/molecular system and hence detect any fast (transient) timescale oscillatory behaviour outside the range of rotating wave approximation (RWA). (ii) The availability of *intense* laser beams ($\approx 10^{12} - 10^{15}$ W cm^{-2}) [4, 5] and their interaction with atomic systems have shown interesting results, such as enhanced dispersion and positive atomic inversion [6]-[8].

In the present article, motivated by points (i) and (ii) above, we review and present governing model Bloch equations in two specific cases:

(a) A single 2-level atom in interaction with a broadband squeezed vacuum reservoir.

(b) A bistable model of a collection of (identical) 2-level atoms in a ring cavity and subject to an additional intense laser field.

II- Study of single 2-level atom in a broadband squeezed vacuum outside RWA

The Hamiltonian for a single 2-level atom with transition frequency ω_0 interacting with the quantised radiation field taken in dipole approximation and without the RWA can be put in the form [9]

$$H = \frac{1}{2}\hbar\,\omega_0\,\sigma_z + \sum_{\underline{\mathbf{k}},\lambda} \hbar\,\omega_k\, a^{\dagger}_{\underline{\mathbf{k}},\lambda}\, a_{\underline{\mathbf{k}},\lambda} - \underline{p}\cdot\underline{\mathbf{e}}\;. \tag{1}$$

Operators $\sigma_{x,y,z}$ are Pauli spin operators while $\underline{\mathbf{k}},\lambda$ label modes of the quantised radiation field: $\omega_k = ck$. The dipole operator $\underline{p} = p\sigma_x\hat{\underline{u}} = p(\sigma_+ + \sigma_-)\hat{\underline{u}}$; p is the matrix element and $\hat{\underline{u}}$ the direction of $\underline{p}$.

The total field operator $\underline{\mathbf{e}}$, quantised in a large box of volume V, is

$$\underline{\mathbf{e}}(t) = i\sum_{\underline{\mathbf{k}},\lambda} \underline{g}_{\underline{\mathbf{k}},\lambda}\,\left(a_{\underline{\mathbf{k}},\lambda}(t) - a^{\dagger}_{\underline{\mathbf{k}},\lambda}(t)\right) \tag{2}$$

and the vector coupling $\underline{g}_{\underline{\mathbf{k}},\lambda} = (2\pi\hbar\omega_k V^{-1})^{1/2}\,\hat{\underline{\epsilon}}_{\underline{\mathbf{k}},\lambda}$: the $\hat{\underline{\epsilon}}_{\underline{\mathbf{k}},\lambda}$ are the unit polarisation vectors.

In the RWA, the operator $p\sigma_+$ couples only to the positive frequency part of the field operator $a_{\underline{\mathbf{k}},\lambda}$ while $p\sigma_-$ couples only to the negative frequency part of the field operator $a^\dagger_{\underline{\mathbf{k}},\lambda}$.

The Su(2) Lie algebra for the atomic spin-$\frac{1}{2}$ operators is

$$\begin{aligned} [\sigma_+, \sigma_-] &= \sigma_z \quad , \quad [\sigma_z, \sigma_\pm] = \pm 2\sigma_\pm \\ [\sigma_+, \sigma_-]_+ &= 1 \quad , \quad [\sigma_\pm, \sigma_z]_+ = 0. \end{aligned} \tag{3}$$

While the field boson operators obey,

$$\left[a_{\underline{\mathbf{k}},\lambda}, a^\dagger_{\underline{\mathbf{k}}',\lambda'}\right] = \delta_{\underline{\mathbf{k}},\underline{\mathbf{k}}'}\delta_{\lambda,\lambda'}. \tag{4}$$

Similarly the spin operators and field operators commute at the same time.

According to the Hamiltonian (1), Heisenberg's equation of motion for the atomic operators in the normal ordering prescription [9] are ($\underline{p} = p\underline{\mathbf{u}}$)

$$\dot{\sigma}_z(t) = -2i\hbar^{-1}\underline{p}.[\underline{\mathbf{e}}^-(t)\,(\sigma_-(t) - \sigma_+(t)) + (\sigma_-(t) - \sigma_+(t))\,\underline{\mathbf{e}}^+(t)] \tag{5}$$

$$\dot{\sigma}_-(t) = -i\omega_0\sigma_-(t) - i\hbar^{-1}\underline{p}.[\underline{\mathbf{e}}^-(t)\sigma_z(t) + \sigma_z(t)\underline{\mathbf{e}}^+(t)] = \left(\dot{\sigma}_+(t)\right)^\dagger. \tag{6}$$

Here the total field operator $\underline{\mathbf{e}}(t) = \underline{\mathbf{e}}^+(t) + \underline{\mathbf{e}}^-(t)$ where $\underline{\mathbf{e}}^\pm(t)$ are the positive and negative frequency parts and [10, 11],

$$\underline{\mathbf{e}}(t)^\pm = \underline{\mathbf{e}}^\pm_o(t) + \underline{\mathbf{e}}^\pm_{RR}(t) \tag{7}$$

where the free fields,

$$\underline{\mathbf{e}}^\pm_0(t) = \pm\, i\sum_{\underline{\mathbf{k}},\lambda} \underline{g}_{\underline{\mathbf{k}},\lambda} \left\{ \begin{array}{c} a_{\underline{\mathbf{k}},\lambda}(0) \\ a^\dagger_{\underline{\mathbf{k}},\lambda}(0) \end{array} \right\} e^{\mp i\omega_k t} \tag{8}$$

and the radiation reaction fields,

$$\underline{\mathbf{e}}^\pm_{RR}(t) = \pm\, i\frac{2}{3}(\frac{\omega_0}{c})^3\underline{p}\sigma_\mp(t) \pm \frac{2}{3\pi}(\frac{\omega_0}{c})^3\underline{p}\ln(\omega_c\omega_0^{-1})[\sigma_+(t) - \sigma_-(t)]\ ; \tag{9}$$

ω_c is the cut-off frequency for ω and can be taken as the Compton frequency.

Using the forms (7-9) into (5), (6) with the use of the algebraic commutation relations (3) we get the coupled set of linear operator equations

$$\begin{aligned} \dot{\sigma_z}(t) &= -\gamma(1+\sigma_z(t)) - 2\hbar^{-1}\underline{p}.\sum_{\underline{\mathbf{k}},\lambda} \underline{g}_{\underline{\mathbf{k}},\lambda}[a^{\dagger}_{\underline{\mathbf{k}},\lambda}(0)\,\sigma_-(t)\,e^{i\omega_k t} \\ &\quad - \sigma_-(t)\,a_{\underline{\mathbf{k}},\lambda}(0)e^{-i\omega_k t} + h.c.] \end{aligned} \tag{10}$$

$$\begin{aligned} \dot{\sigma_-}(t) &= (-\frac{1}{2}\gamma - i\omega_0')\sigma_-(t) + (\frac{1}{2}\gamma - i\Delta_0)\sigma_+(t) \\ &\quad - \hbar^{-1}\underline{p}.\sum_{\underline{\mathbf{k}},\lambda} \underline{g}_{\underline{\mathbf{k}},\lambda}[a^{\dagger}_{\underline{\mathbf{k}},\lambda}(0)\,\sigma_z(t)\,e^{i\omega_k t} - \sigma_z(t)\,a_{\underline{\mathbf{k}},\lambda}(0)e^{-i\omega_k t}] \\ &= (\dot{\sigma_+}(t))^+ \end{aligned} \tag{11}$$

where $\gamma = 4p^2\omega_o^3/(3\hbar c^3)$ is the Einstein A-(damping) coefficient, ω_0' is the atomic frequency ω_0 shifted by the *ordinary* vacuum ($\omega_0' = \omega_0 - \Delta_0$; $\Delta_0 = \gamma\pi^{-1}\ln(\omega_c/\omega_0)$ [8, 9. 11] and $\Delta_0\omega_o^{-1} = O(\gamma\omega_0^{-1})$).

Now, we derive the equation for the mean atomic inversion $r_3(t) = < \sigma_z(t) >$ where $|>$ is the initial state of the combined system (atom+squeezed vacuum field). The formal integration of the operator equation (11) for $\sigma_-(t)$ gives

$$\begin{aligned} \sigma_-(t) &= \sigma_-(0)e^{-(\frac{1}{2}\gamma+i\omega_0')t} + \int_0^t e^{-(\frac{1}{2}\gamma+i\omega_0')(t-t')}\{(\frac{1}{2}\gamma - i\Delta_0)\sigma_+(t') \\ &\quad -\hbar^{-1}\underline{p}.\sum_{\underline{\mathbf{k}},\lambda} \underline{g}_{\underline{\mathbf{k}},\lambda}[a^{\dagger}_{\underline{\mathbf{k}},\lambda}(0)\sigma_z(t')\,e^{i\omega_k t'} - \sigma_z(t')a_{\underline{\mathbf{k}},\lambda}(0)e^{-i\omega_k t'}]\}dt' \\ &= (\sigma_+(t))^+. \end{aligned} \tag{12}$$

Inserting this result into equ.(10) we get,

$$\dot{\sigma_z}(t) = -\gamma\,(1+\sigma_z(t)) + \mathcal{A} + \mathcal{B} + \mathcal{C}. \tag{13}$$

The t-dependent operators $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are

$$\begin{aligned} \mathcal{A} &= -2\hbar^{-1}\underline{p}.\sum_{\underline{\mathbf{k}},\lambda} \underline{g}_{\underline{\mathbf{k}},\lambda}\{a^{\dagger}_{\underline{\mathbf{k}},\lambda}(0)\,[\sigma_-(0)\,e^{-\Gamma_o t} \\ &\quad - \sigma_+(0)e^{-\Gamma_o^* t}]e^{i\omega_k t} + h.c.\} \end{aligned} \tag{14a}$$

$$\begin{aligned} \mathcal{B} &= -2\hbar^{-1}\underline{p}.\sum_{\underline{\mathbf{k}},\lambda} \underline{g}_{\underline{\mathbf{k}},\lambda}\int_0^t \{a^{\dagger}_{\underline{\mathbf{k}},\lambda}(0)\,[(\frac{1}{2}\gamma - i\Delta_0)\,\sigma_+(t')e^{-\Gamma_o(t-t')} \\ &\quad -(\frac{1}{2}\gamma + i\Delta_0)\,\sigma_-(t')e^{-\Gamma_o^*(t-t')}]e^{i\omega_k t} + h.c.\}dt' \end{aligned} \tag{14b}$$

and

$$C = -2\hbar^{-2}\underline{p}\,\underline{p}\sum_{\underline{\mathbf{k}},\lambda}\sum_{\underline{\mathbf{k}}',\lambda'} \underline{g}_{\mathbf{k},\lambda}\,\underline{g}_{\mathbf{k}',\lambda'}\cdot$$

$$\int_0^t \{\, a^{\dagger}_{\underline{\mathbf{k}},\lambda}(0)\sigma_z(t')\,a_{\underline{\mathbf{k}}',\lambda'}(0)\,e^{-\Gamma_o(t-t')+i(\omega_k t-\omega_{k'}t')}$$

$$+\,a^{\dagger}_{\underline{\mathbf{k}}',\lambda'}(0)\sigma_z(t')\,a_{\underline{\mathbf{k}},\lambda}(0)e^{-\Gamma_o(t-t')-i(\omega_k t-\omega_{k'}t')}$$

$$-\,a^{\dagger}_{\underline{\mathbf{k}},\lambda}(0)\,a^{\dagger}_{\underline{\mathbf{k}}',\lambda'}(0)\,\sigma_z(t')\,e^{-\Gamma_o(t-t')+i(\omega_k t+\omega_{k'}t')}$$

$$-\,\sigma_z(t')\,a_{\underline{\mathbf{k}}',\lambda'}(0)\,a_{\underline{\mathbf{k}},\lambda}(0)\,e^{-\Gamma_o(t-t')-i(\omega_k t+\omega_{k'}t')} + h.c.\}dt' \qquad (14c)$$

where $\Gamma_0 = \frac{1}{2}\gamma + i\omega_0'$. The operator expressions in (14) are complete and their expectation values are to be taken with respect to the initial combined (squeezed vacuum field+atom) state. Within the RWA (i.e. dropping the terms in $a^{\dagger}_{\underline{\mathbf{k}},\lambda}\sigma_+$ and $\sigma_- a_{\underline{\mathbf{k}},\lambda}$ in the Hamiltonian (1)), the mean atomic inversion shows a purely decaying behaviour [11] just as in the case of broadband thermal field case [12], (also see[13]).

As in [11] we assume the squeezed vacuum field is characterized by the following relations

$$\sum_{\lambda,\lambda'} \hat{\underline{\epsilon}}_{\mathbf{k},\lambda}\,\hat{\underline{\epsilon}}_{\mathbf{k}',\lambda'} < a^{\dagger}_{\underline{\mathbf{k}},\lambda}\,a_{\underline{\mathbf{k}}',\lambda'} >= N_k(k\,k')^{-1}\delta(k-k')\,(\mathbf{U}-\hat{\mathbf{k}}\,\hat{\mathbf{k}}) \qquad (15\text{ a})$$

$$\sum_{\lambda,\lambda'} \hat{\underline{\epsilon}}_{\mathbf{k},\lambda}\,\hat{\underline{\epsilon}}_{\mathbf{k}',\lambda'} < a^{\dagger}_{\underline{\mathbf{k}},\lambda}\,a^{\dagger}_{\underline{\mathbf{k}}',\lambda'} > = M^{*}_{k,k'}(k\,k')^{-1}\delta(k+k'-2k_p)\,(\mathbf{U}-\hat{\mathbf{k}}\,\hat{\mathbf{k}}) \qquad (15\text{ b})$$

$$< a_{\underline{\mathbf{k}},\lambda} > \;=\; < a^{\dagger}_{\underline{\mathbf{k}},\lambda} > \;=\; 0\,. \qquad (15\text{ c})$$

The notations are: $\mathbf{U}$ is the unit tensor and $\hat{\mathbf{k}}$ is the direction of the wave vector $\mathbf{k} = k\hat{\mathbf{k}}$ with dyadic $\hat{\mathbf{k}}\,\hat{\mathbf{k}}$; while k is the mode wavenumber, $\omega = ck$ is the mode frequency and, in particular, $2\,\omega_p \equiv (2\,k_p)c$ is the frequency of the field pumping the squeezed vacuum.

The squeezed vacuum field parameters: the real N_k and the complex number $M_{k,k'} \equiv M_k$ (for simplicity) are the average photon number at wave numbers k and degree of squeezing respectively, with $|\,M_k\,|^2 = N_k(N_k+1)$ for minimum uncertainty [9. and refs therein].

Now, with the relations (15) for the squeezed vacuum field we notice the following regarding the operator expressions on the right side of equation (13):

(i) $<\mathcal{A}>=0$ by equ(15 c),

(ii) The terms in $\mathcal{B}$ will give rise to smaller terms $O(\gamma^2/\omega_o^2)$ [9, 12] upon substituting for $\sigma_\pm(t)$ by its integral form, equ(13). We drop these small order terms as we keep terms $O(\gamma/\omega_o)$ outside the RWA.

(iii) For the expression in C we use the unequal-time free field-matter operators [12] within Markov approximation,

$$\left[e_o^{\mp}(t), \sigma(t')\right] = 0 \qquad ; t > t' \tag{16}$$

with $\sigma(t')$ is any atomic operator and the statistical field-atom de-correlation procedure (see [11, 12] for detail) and finally reach the generalised rate equation for the atomic inversion,

$$\dot{r}_3(t) = -\gamma - \gamma\left[(1+2N) + 2\mid M\mid \cos(2\omega_0' t - \phi)\right] r_3(t) \tag{17}$$

where N and $M = \mid M \mid e^{i\phi}$ are the values of N_k and M_k at the resonant mode $k = \omega_o/c$, and ϕ is the squeezed vacuum phase. Within the RWA, the highly oscillatory term in $\cos 2\omega_o' t$ is ignored and hence (17) reduces to the familiar rate equation in the thermal field case [12]. Thus the new feature in (17) is related to the squeezed vacuum parameters M outside the RWA.

The formal solution of (17) is,

$$\begin{aligned} r_3(t) &= e^{-\gamma(1+2N)t} e^{-b\sin(2\omega_0' t-\phi)}\{r_3(0)e^{-b\sin(\phi)} \\ &\quad -\gamma \int_0^t dt' e^{\gamma(1+2N)t' + b\sin(2\omega_0' t-\phi)}\}. \end{aligned} \tag{18}$$

where $b \equiv \gamma \mid M \mid /\omega_0'$ is the correction parameter outside the RWA. The integral in (18) can be expressed in terms of the modified Bessel functions $I_n(b)$. For long time (steady state) we get

$$r_3(\infty) = -\gamma(2\omega_0')^{-1} e^{-b\sin 2\omega_0' t}\left\{\frac{1}{\alpha} I_0(b) + 2\, Re \sum_{n=1}^{\infty} \frac{(\alpha - i n) I_n(b)}{(\alpha^2 + n^2)} e^{in(2\omega_0' t - \frac{\pi}{2})}\right\} \tag{19}$$

in which $\alpha \equiv \gamma(1+2N)/2\omega_0'$. The factor $e^{-b\sin 2\omega_0' t}$ also has an expression in terms of the $I_n(b)$ and the *non-oscillatory component* easily proves to be

$$(r_3(\infty))_o = -\frac{1}{1+2N} I_0^2(b) = -\frac{1}{1+2N}\left(1+\frac{1}{2}\left(\frac{\gamma \mid M \mid}{\omega_0'}\right)^2 + ...\right). \quad (20)$$

The transient oscillatory behaviour of $r_3(t)$ outside the RWA according to (18) is shown in fig(1) as compared with the purely damped behaviour within the RWA. The steady oscillatory behaviour (of period π/ω_o') of $r_3(\infty)$, equ(19), is shown in fig(2) for the parameter (γ/ω_o')=2(10^{-4}). The rest of the generalised Bloch equations,

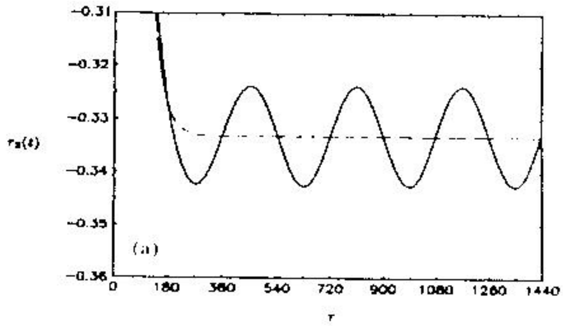

Figure 1: The mean atomic inversion against the normalised time $\tau = 2\omega_o' t$ for $(\gamma/2\omega_o')=10^{-2}$ for initially inverted atom ($r_3(0) = 1$) with squeezed vacuum parameters $N = 1$, $|M| = \sqrt{2}$ and $\phi = 0$. The full curve is the solution (18) outside the RWA and the broken curve is the RWA solution.

namely the equations for the polarisation components $r_\pm(t) =< \sigma_\pm(t) >$, can be derived by a similar procedure and have the following forms (all shifts are dropped for simplicity [9]),

$$\begin{aligned} r_-^{\cdot}(t) &= -(\Gamma + i\omega_o')r_-(t) - \gamma\left[\left(M_1 e^{-2i\omega_p t} - M_3^* e^{2i\omega_p t}\right) r_+(t) + c.c.\right] \\ &= \left(r_+^{\cdot}(t)\right)^*. \end{aligned} \quad (21)$$

The system (21) has harmonic coefficients and its solutions contain all harmonics at $(2\omega_p)$. Analysis of the spectrum using (21) was given in [9]. Generalisation to the case of two-cooperative atoms in a similar situation [14] shows that the amplitude of the steady oscillation is larger compared with the single atom case.

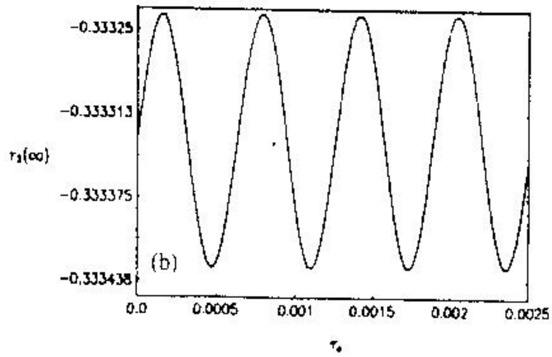

Figure 2: The steady oscillatory atomic inversion $r_3(\infty)$, equ(19), against the normalised time $\tau_o = \gamma t$ for values $N = 1$, $|M| = \sqrt{2}$ and $\phi = 0$, $(\gamma/2\omega_o')=10^{-4}$.

III- Multi-stable behaviour with field-dependent relaxation in a ring cavity

Here we consider the model of a collection of 2-level atoms subject to an intense field (as part of its environmental reservoir) and placed inside a driven ring cavity (see fig(3)). The intense field (b) driving the atoms is introduced at the mirror M_2 so that no interference or interaction takes place between the coherent input field a_{in} at mirror M_1 or the cavity field and the intense field b at M_2

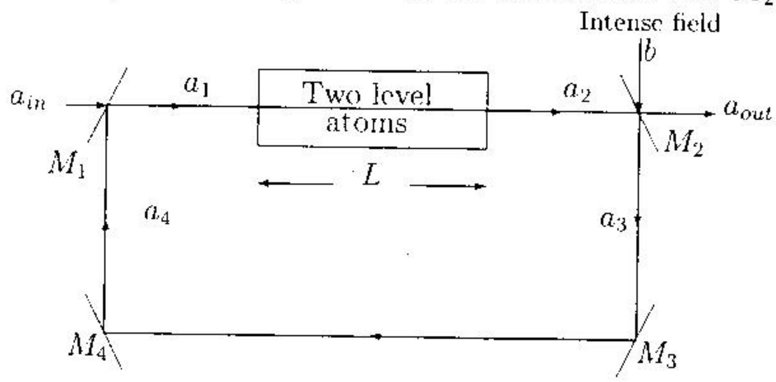

Figure 3: Schematic representation of a ring cavity containing 2-level atoms intensely driven by field b at mirror M_2 while the cavity is driven by a coherent field input a_{in} at mirror M_1. a_{out} is the output field at M_2; T and R are the transmissivity and refractivity of the mirrors M_1, M_2. L is the cavity length.

The generalised model Bloch equations for the above system for the averaged atomic polarisation components $J_\pm$ and the inversion J_z within the RWA are of the form,

$$\begin{aligned}\frac{\partial J_-}{\partial t} &= -(\frac{\eta}{2}+i\Delta)\,J_- + \eta_o e^{i\phi_f} + (2g\alpha - i\Omega_o\, e^{i\phi_f})\,J_z \\ &= \left(\frac{\partial J_+}{\partial t}\right)^* \end{aligned} \tag{22}$$

$$\begin{aligned}\frac{\partial J_z}{\partial t} &= -\eta(\frac{1}{2}+J_z) + \eta_o\,(e^{i\phi_f}\,J_+ + e^{-i\phi_f}\,J_-) \\ &\quad -(g\alpha + i\frac{\Omega_o}{2}e^{i\phi_f})\,J_+ - (g\alpha^* - i\frac{\Omega_o}{2}e^{-i\phi_f})\,J_-. \end{aligned} \tag{23}$$

The notations are: Ω_o is the Rabi frequency of the intense laser field b. $\eta_o = 3\gamma\lambda/4\sqrt{1+\delta_1^2}$, $\Delta = \omega_o - \omega_L$, ω_o is the atomic frequency, ω_L is the laser field frequency, $\eta = \gamma(1+3\lambda\delta_1/\sqrt{1+\delta_1^2})$, $\delta_1 = (\Delta/\Omega_o) < 1$, $\lambda = (\Omega_o/\omega_L) < 1$, $\beta = \frac{1}{2}\eta + i\Delta$ and ϕ_f is the phase of the intense laser field b, α is the cavity field and g is the coupling constant between the cavity field and the atoms.

Note that in the absence of the cavity ($\alpha = 0$) and for $\Delta = 0$, equations (22) and (23) reduce to those derived in [6] at exact resonance. Full derivation of equations (22) and (23) using operator reaction field theory are outside the scope of the present article and will be presented elsewhere [15]. The boundary conditions according to the scheme in fig(3) are: At mirror M_1:

$$a_1 = \sqrt{T}\,a_{in} + \sqrt{R}\,a_4 \tag{24}$$

From mirror M_1 to mirror M_2;

$$a_2 = a_1 + i\,\frac{gL}{c}J_- \tag{25a}$$

$$= (\sqrt{T}\,a_{in} + \sqrt{R}\,a_4) + i\,\frac{gL}{c}J \tag{25b}$$

The output at mirror M_2:

$$a_{out} = \sqrt{T}\,a_2 + \sqrt{R}\,b \tag{26}$$

$$= T\,a_{in} + \sqrt{T\,R}\,a_4 + i\,\sqrt{T}\,\frac{g\,L}{c}J_- + \sqrt{R}\,b \tag{27}$$

where we used (25b) in (27).

From mirror M_2 to mirror M_3:

$$a_3 = \sqrt{R}\, a_2 + \sqrt{T}\, b \tag{28}$$

Substituting eq(25b) into eq(28) we get;

$$a_3 = R\, a_4 + \sqrt{T\, R}\, a_{in} + i\, \sqrt{R}\, \frac{g\, L}{c} J_- + \sqrt{T}\, b \tag{29}$$

From mirror M_2 through mirrors M_3 and M_4 to mirror M_1;

$$a_4 = a_3 e^{i\phi_o} \tag{30}$$

where ϕ_o is a phase shift dependent on the cavity detuning. Using (29) into (30) we get

$$a_4 = e^{i\phi_o} \left(R\, a_4 + \sqrt{T\, R}\, a_{in} + i\, \sqrt{R}\, \frac{g L}{c} J_- + \sqrt{T}\, b \right). \tag{31}$$

Putting $\phi_o = \theta\, T$ where $\theta =$ is the normalised cavity detuning we get from (31);

$$a_4 = e^{i\theta T} \left(\sqrt{TR} a_{in} + i \sqrt{R} \frac{g L}{c} J_- + \sqrt{T} b \right) (1 - R e^{i\theta T})^{-1}. \tag{32}$$

Using (32) into (27) we get

$$\left(1 - R e^{i\theta T}\right) a_{out} = T a_{in} + \ i\, \sqrt{T}\, \frac{g\, L}{c} J_- + \sqrt{R}\, b\, \left[(2T - 1) e^{i\theta T} + 1\right]. \tag{33}$$

Up to $O(T)$ and using the steady solution for J_- (obtained from equs(22-23) by putting $\frac{\partial}{\partial t} J_{z,\pm} = 0$) one gets (after some manipulation) the generalised input-output steady state relation.

$$|y| = |(1 - i\theta)\, |x|\, e^{i\phi} + \bar{\Omega}_o\, \bar{\lambda}_2\, (2 - i\theta) - 2\, C \beta^{-1} \left[\bar{\eta}_o + (|x|\, e^{i\phi} - i\bar{\Omega}_o)\, A_1 / B_1\right]| \tag{34}$$

where

$$\begin{aligned} A_1 &= -\frac{1}{2} \bar{\eta}\, |\bar{\beta}|^2 + \bar{\eta}_o^2 \bar{\eta} - \frac{\bar{\eta}_o}{2} \left[\left(\bar{\beta}^* e^{-i\phi} + c.c. \right) |x| - 2\, \bar{\Omega}_o \bar{\Delta} \right] \\ B_1 &= \bar{\eta}\, |\bar{\beta}|^2 - \bar{\eta}_o \left[\left(\bar{\beta}^* e^{+i\phi} + c.c. \right) |x| - 2 \bar{\Omega}_o \Delta \right] \\ &\quad + \frac{\bar{\eta}}{2} \left(|x|^2 - \bar{\Omega}_o^2 \right) - 2\, \bar{\Delta}\, \bar{\Omega}_o\, |x|\, cos(\phi) \end{aligned} \tag{35}$$

where $\bar{\eta}_o = \eta_o/\gamma$, $\bar{\eta} = \eta/\gamma$, $\bar{\beta} = \beta/\gamma$, $\bar{\Delta} = \Delta/\gamma$, $C = g^2 L n_o/(\gamma c T)$ is the cooperativity parameter, $x = 2g a_{out}/(\gamma\sqrt{T}) = |x| e^{i\phi_x}$ is the scaled output field, $y = 2g a_{in}/(\gamma\sqrt{T})$ is the scaled input field at mirror M_1, $\bar{\Omega}_o = \Omega_o/(\gamma\sqrt{T}) \equiv p\hbar^{-1}b/(\gamma\sqrt{T})$ is the scaled *intense* field at mirror M_2, $\phi = \phi_x - \phi_f$ is the relative phase of the output field x with respect to the phase of the intense field $\bar{\Omega}_o$ and $\bar{\lambda}_2 = 2g\sqrt{R}/(p\hbar^{-1})$ is essentially the ratio of the two coupling constants between both fields y, $\bar{\Omega}_o$ and the atoms. Putting $\bar{\Omega}_o = 0$ in equ.(34) it reduces to the usual form [16] in the weak (independent-field) damping case.

The relation (34) is plotted in figs(4-6) for the given parameters data. In the resonant case ($\Delta = \theta = 0$), the system shows tri-bistability for the cases of phase value $\phi = 0, -\pi/2$ (but not for $\phi = \pi, \pi/2$), figs(4). In the off-resonance case where $\Delta\theta > 0$ similar behaviour occurs for small values of Δ, θ but for larger $\bar{\Delta} = \bar{\theta} = 5$ bistability occurs for $\phi = \pi/2$ while for $\bar{\Delta} = \bar{\theta} = -5$ a tri-bistability occurs for $\phi = -\pi/2$ (fig.5). On the other hand, for $\bar{\Delta}\bar{\theta} < 0$ with small values $\bar{\Delta} = -\bar{\theta} = \pm 0.1$ tri-bistability occurs for $\phi = 0, -\pi/2$. For larger values of $\bar{\Delta} = -\bar{\theta} = \pm 5$ the tri-bistability occurs only for phase value $\phi = -\pi/2$ (fig.6).

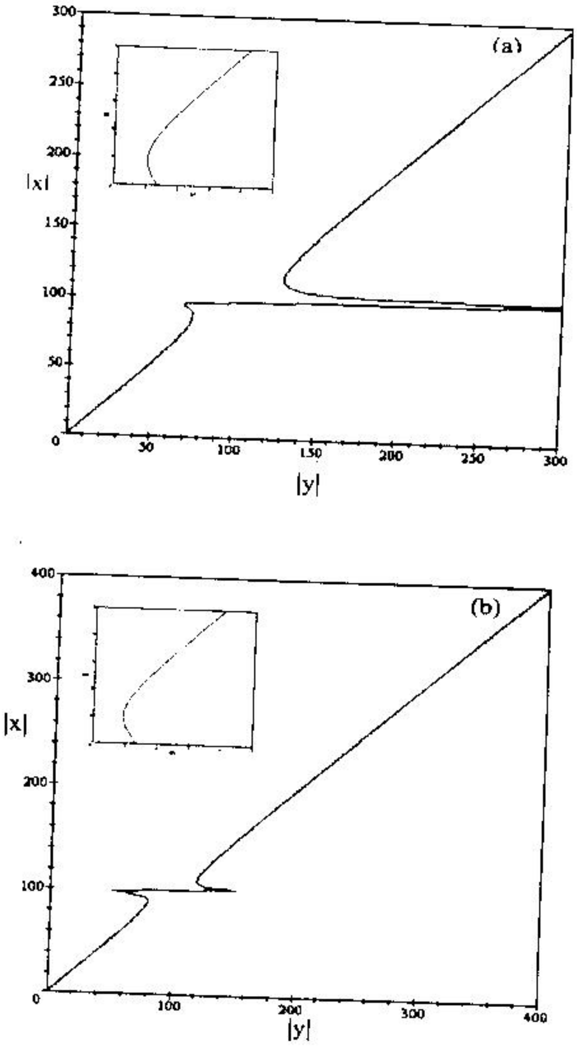

Figure 4: $|x|$ - $|y|$ plot for $\Omega_o = C = 100$, $\delta = \theta = 0$ (pure resonant), $\lambda = 0.01$, $\lambda_2 = T = 0.01$ (a)$\phi = 0$ and (b)$\phi = -\pi/2$. The enset shows the part of the curve for $|y| \in [0, 5]$

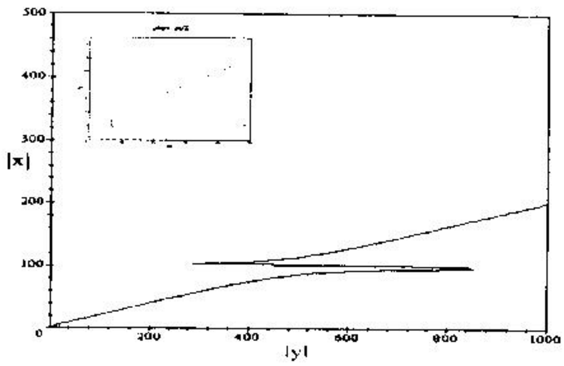

Figure 5: $|x| - |y|$ plot for $\Omega_o = C = 100$, $\delta = \theta = -5$, $\lambda = 0.01$, $\lambda_2 = T = 0.01$ and $\phi = -\pi/2$. The enset shows the part of the curve for $|y| \in [0, 50]$

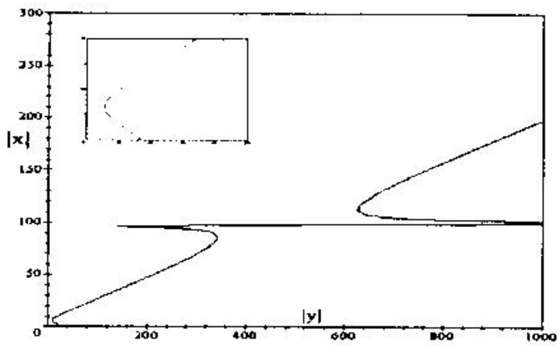

Figure 6: $|x| - |y|$ plot for $\Omega_o = C = 100$, $\delta = -\theta = 5$, $\lambda = 0.01$, $\lambda_2 = T = 0.01$ and $\phi = -\pi/2$. The enset shows the part of the curve for $|y| \in [0, 100]$

IV- Summary

Motivated by the availability of the femtosecond time resolution technique and the high power laser sources we have given two examples of quantum mathematical modelling which predict new features. Specifically,

(a) A single 2-level atom bathed in a phase-sensitive (squeezed vacuum) reservoir is treated outside the RWA (i.e. by keeping the fast oscillatory terms). The mean atomic inversion exhibits both transient and steady oscillation that could be detected in principle by the technique in [3].

(b) A collection of identical 2-level atoms subject to an *intense* field (as part of its environmental reservoir) and placed inside a ring cavity is treated within the RWA to show field-dependent damping coefficients. The effect of such coefficients causes the usual bistable curve to become tri-stable curve for the input-output field relationship. The availability of intense laser fields stimulates the appetite for such investigation. Further investigation concerning the asymmetry in the absorption/dispersion spectra will be given in [15]

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