JKAU: Sci., **vol. 15,** pp. 33-42 (1423 A.H. / 2003 A.D.)

A Fast Algorithm to Solve The Poisson Equation in The Complex Plane

DAOUD MASHAT
Dept. of Mathematics, King Abdulaziz University, Jeddah, Saudi Arabia

ABSTRACT. Fast algorithm for the accurate evaluation of some integral operator that arise in the context of solving certain partial differential equations within the unit circle in the complex plane are presented. It is based on some recursive relations in the Fourier space and the FFT (Fast Fourier Transform), and have theoretical computational complexity of the order $O(\log N)$ per point, where N^2 is the total number of grid points.

1. Introduction

Many problems in applied mathematics require the evaluation of the singular integral transform

$$Lh(\sigma) = \int\int_{B(0,1)} h(\zeta)\log|\zeta - \sigma|\, d\xi d\eta,\ \ \zeta = \xi + i\eta \tag{1.1}$$

of a complex valued function h defined on $B(0, 1) = \{\sigma : |\sigma| < 1\}$, for example[2], the general solution of the Poisson equation

$$\Delta\, \upsilon = h \tag{1.2}$$

in the unit circle is given by

$$\upsilon(\sigma) = \frac{1}{2\pi} L(\sigma) \tag{1.3}$$

The method presented takes into account the convolution nature of this integral and some of the properties of such convolution integrals in Fourier space, the set of all complex continuous functions defined on $[0, 2\pi]$[1]. This process leads to a recursive algorithm in Fourier space that divides the entire domain into a collection of circular regions and expands the integral in Fourier series with radius dependent Fourier coefficients. A set of exact recursive relations involve appropriate scaling of one-dimensional integrals in circular regions, which significantly improves the computational complexity. The desired integrals at all N^2 grid points are then easily obtained from the Fourier coefficients by the FFT (fast Fourier transform). The process of evaluation of these integrals has thus been optimized in this paper giving a net operation count of the $O(\ln N)$ per point. This algorithm has the added advantage of working in place, meaning that no additional memory storage is required beyond that of the initial data.

This paper is laid out as follows. In section 2 we present the mathematical foundation of fast algorithm for rapid evaluation of $Lh(\sigma)$ within the unit circle. In section 3, the formal description of the fast algorithm is presented. Finally, we summarize and conclude in section 4.

2. Evaluation of *L*-Operator

In this section we develop the theory needed to construct an efficient algorithm for evaluation of the *L*-operator.

Theorem 1.2. If $Lh(\sigma)$ exists in the unit disk, and $h(\sigma = re^{i\alpha}) = \sum_{n=-\infty}^{\infty} h_n(r)e^{in\alpha}$, then the n^{th} Fourier Coefficients $Lh(\sigma = re^{i\alpha})$ is given for $\sigma \neq 0$ by

$$\begin{cases} \frac{\pi r^n}{n}\int_0^r \\ varphi^{(1-n)} h_n(\varphi)d\varphi, & n \leq 1, \\ 2\pi\int_r^1 \{\frac{1}{2}\varphi h_0(\varphi)\log(\varphi) + \frac{i}{2}\varphi H_0(\varphi)\}d\varphi, & n = 0, \\ -\frac{\pi r^n}{n}\int_r^1 \varphi^{(1-n)} h_n(\varphi)d\varphi, & n \geq 1. \end{cases} \tag{2.1}$$

where $\alpha h(\sigma = re^{i\alpha}) = \sum_{n=-\infty}^{\infty} H_n(r)e^{in\alpha}$.

Proof. For $\sigma \neq 0$ we have

$$\begin{aligned} &\frac{1}{2\pi}\int_0^{2\pi} e^{-in\alpha} \lim_{\in\to 0}\int\int_{B(0,1)-B(\sigma,\in)} h(\zeta)\log|\zeta-\sigma|\, d\xi d\eta \\ &= \lim_{\in\to 0}[\int\int_{\Omega_1} Q(r,\zeta)h(\zeta)d\xi d\eta] + \lim_{\in\to 0}[\int\int_{\Omega_1} Q(r,\zeta)h(\zeta)d\xi d\eta] \end{aligned} \tag{2.2}$$

$$= \begin{cases} \dfrac{r^n}{2n\xi^n}, & n \le -1, |\zeta| < |\sigma|; \\ 0, & n \ge 0, |\zeta| < |\sigma|; \\ \dfrac{1}{2}\log\zeta, & n = 0, |\zeta| > |\sigma|; \\ 0, & n \le -1, |\zeta| > |\sigma|; \\ -\dfrac{r^n}{2n\xi^n}, & n \ge 1, |\zeta| > |\sigma|. \end{cases} \tag{2.3}$$

where $Q(r,\zeta) = \frac{1}{2\pi}\int_0^{2\pi} e^{-in\alpha} \log|\zeta - \sigma|\, d\alpha$, $\Omega_1 = \{\zeta : |\zeta| \le r - \in\}$,, and $\Omega_2 = \{\zeta : r + \in \le |\zeta| \le 1\}$, (see Lemma A.4).

If we set $\zeta = \rho e^{i\theta}$, we get

$$I_1 + I_2 = \begin{cases} \int_0^r\int_0^{2\pi} 0h(\rho e^{i\theta})\dfrac{r^n}{2n\rho^n e^{in\theta}}\rho d\theta d\rho, & n \le -1; \\ \dfrac{1}{2}\int_r^1\int_0^{2\pi} h(\rho e^{i\theta})\log(\rho e^{i\theta})\rho d\theta d\rho, & n = 0, \\ \int_r^1\int_0^{2\pi} -h(\rho e^{i\theta} \dfrac{r^n}{2n\rho^n e^{in\theta}}\rho d\theta d\rho, & n \ge 1, \end{cases} \tag{2.4}$$

$$= \begin{cases} \dfrac{\pi r^n}{n}\int_0^r \rho^{(1-n)} h_n(\rho)d\rho, & n \le -1; \\ 2\pi\int_r^1 \dfrac{1}{2}\rho\log\rho h_0(\rho)d\rho + \dfrac{2\pi i}{2}\int_r^1 \rho H_0(\rho)d\rho, & = 0; \\ -\dfrac{\pi r^n}{n}\int_r^1 \rho^{(1-n)} h_n(\rho)d\rho, & n \ge 1, \end{cases} \tag{2.5}$$

where $H_0(\rho)$ is the constant term in the Fourier Series Coefficients of $\theta h(\rho e^{i\theta})$.

Theorem 2.2. If $\sigma = 0$, then

$$Lh(0) = \int\int_{B(0,1)} h(\zeta)\log|\zeta|\, d\xi d\eta = 2\pi \lim_{\in \to 0}\int_{\in}^1 \rho h_0(\rho)\log\rho d\rho. \tag{2.6}$$

Proof. Since

$$\begin{aligned} Lh(0) &= \int\int_{B(0,1)-B(0\in)} h(\zeta)\log|\zeta|\, d\xi d\eta = \int_{\in}^1\int_0^{2\pi} h(\rho e^{i\theta})\log|\rho e^{i\theta}|\,\rho d\theta d\rho \\ &= \int_{\in}^1 \rho\log\rho\int_0^{2\pi} h(\rho e^{i\theta})d\theta d\rho \\ &= 2\pi\int_{\in}^1 \rho h_0(\rho)\log\rho d\rho. \end{aligned} \tag{2.7}$$

Therefore

$$\int\int_{B(0,1)} h(\zeta)\log|\zeta|\, d\xi d\eta = 2\pi \lim_{\epsilon\to 0}\int_{\epsilon}^{1} \rho h_0(\rho)\log\rho d\rho. \tag{2.8}$$

If we denote the n[th] Fourier Series Coefficient of $Lh(\sigma = re^{i\alpha})$ by $C_n(r)$ then we get the following corollaries.

Corollary 1.2. It follows directly from (2.1) that

$$C_n(0) = 0 \quad \text{for} \quad n \le 1. \tag{2.9}$$

$$C_n(1) = 0 \quad \text{for} \quad n \ge 0. \tag{2.10}$$

Corollary 2.2. For $r_j > r_i$, if we define

$$C_n^{ij} = \begin{cases} \dfrac{\pi r_j^n}{n}\displaystyle\int_{r_i}^{r_j} \rho^{(1-n)} h_n(\rho)d\rho, & n \le -1; \\ 2\pi\displaystyle\int_{r_i}^{r_j} \{\rho\log(\rho)^{1/2} h_0(\rho) + \rho H_0(\rho)\}d\rho, & n = 0 \\ -\dfrac{\pi r_i^n}{n}\displaystyle\int_{r_i}^{r_j} \rho^{(1-n)} h_n(\rho)d\rho, & n \ge 1. \end{cases} \tag{2.11}$$

then by mathematical induction it follows that

$$C_n(r_j) = (\frac{r_j}{r_i})C_n(r_i) + C_n^{ij}, \quad n \le -1, \tag{2.12}$$

$$C_0(r_i) = C_0^{ij} + C_0(r_j), \tag{2.13}$$

$$C_n(r_i) = (\frac{r_i}{r_j})C_n(r_j) + C_n^{ij}, \quad n \ge 1. \tag{2.14}$$

Corollary 2.3. Let $0 = r_i < r_2 < ... < r_M = 1$, then by mathematical induction it follows that

$$C_n(r_l) = \begin{cases} \displaystyle\sum_{i=2}^{l} (\frac{r_l}{r_i})^n C_n^{i-1,i}, & \text{for } n \le -1 \text{ and } l = 2,3,...,M; \\ \displaystyle\sum_{i=l}^{M-1} C_0^{i,i+1}, & \text{for } n = 0 \text{ and } l = 2,3,...,M-1; \\ \displaystyle\sum_{i=l}^{M-1} (\frac{r_l}{r_i})^n C_n^{i,i+1}, & \text{for } n \ge 1 \text{ and } l = 2,3,...,M-1 \end{cases} \tag{2.15}$$

3. The Fast Algorithm

If we discretized the disk by using $M \times N$ lattice points with M equidistant points in the radial direction and N equidistant points in the circular direction, then we can construct the fast algorithm based on the theory of section 2. The following is a formal description of the algorithm.

Algorithm

Input. The integer M, the number of points in the radial direction; N, the number of points in the circular direction; $h(\sigma = r_l e^{2\pi ik/N})$, $l \in [1, M]$ and $k \in [1, N]$.

Output. $Lh\ (\sigma = re^{2\pi ik/N})$, $l \in [1, M]$ and $k \in [1, N]$. Compute $h_s(r_l)$ for $s \in [-Q, Q]$.

Step 3. From $h(\sigma = r_l e^{2\pi ik/N})$, $l \in [1, M]$ and $k \in [1, N]$. Compute $H_0(r_l)$.

Step 4. For $i \in [2, M-1]$ and $s \in [-Q, Q]$. Compute $C_s^{i,i+1}$ as follows

$$C_s^{i,i+1} = \begin{cases} \dfrac{\pi r_{i+1}^s}{s} \displaystyle\int_{r_i}^{r_i+1} \rho^{(1-s)} h_s(\rho) d\rho, & s \in [Q, -1]; \\ 2\pi \displaystyle\int_{r_i}^{r_i+1} \{\rho \log(\rho)^{1/2} h_0(\rho) + \rho H_0(\rho)\} d\rho, & s = 0; \\ -\dfrac{\pi r_i^s}{s} \displaystyle\int_{r_i}^{r_i+1} \rho^{(1-s)} h_s(\rho) d\rho, & s \in [1, Q]. \end{cases}$$

Step 5. Compute $C_s(r_l)$ for $s \in [-Q, Q]$ and $l \in [1, M]$ as follows

set $C_s(r_M) = 0$ for $s \in [0, Q]$

do $s = 0, 1, ..., Q$

$$C_s(r_l) = C_s^{l,l+1} + \left(\frac{r_l}{r_{l+1}}\right)^s C_s(r_{l-1})$$

enddo

enddo

Set $C_s(r_1) = 0$ for $s \in [-Q, -1]$

do $s = -Q, ..., -1$

do $l = 2, 3, ..., M$

$$C_s(r_l) = C_s^{l-1,l} + \left(\frac{r_l}{r_{l-1}}\right)^s C_s(r_{l-1})$$

enddo

enddo

Step 6. Compute $C_0(r_1)$ as follows

$$C_0(r_1) = 2\pi \lim_{\in \to 0} \int_{\in}^{1} \rho h_0(\rho) \log(\rho) d\rho$$

Step 7. Output

$$Lh(\sigma = r_l e^{2\pi ik/N}) = \sum_{s=-Q}^{Q} C_s(r_l) e^{2\pi ik/N} \quad \text{for } l \in [1, M] \text{ and } k \in [1, N].$$

Stop

Remark 3.1. N must be a power of two.

Remark 3.2. We can use Laguerre polynomials to approximate the integral in step 6.

Conclusions

The present work develops a fast algorithm to evaluate the singular integral operator L in the interior of a unit disk in the complex plane. It is based on computation of the integral from its Fourier coefficients. The recursive relations satisfied by these Fourier coefficients are derived, which are at the heart of the algorithm. The speed up provided by the algorithm is dramatic even for a moderate number of nods in the domain. In actual implementation, the error will arise from finite truncation of Fourier series and approximate evaluation of the one dimensional integral.

References

[1] **Daripa, P.** and **Mashat, D.,** Singular integral transforms and fast numerical algorithms, *Numerical algorithms* **18:** 133-157 (1998).

[2] **Greenberg, M.,** *Application of Greens Functions in Science and Engineering,* Prentice-Hall, New York (1971).

Appendix

Lemma A.1. If $\zeta \neq \sigma$., then

$$\frac{\frac{d}{d\zeta}|\zeta-\sigma|}{|\zeta-\sigma|}=\frac{1}{2(\zeta-\sigma)}, \tag{A.1}$$

Proof. Let ζ = ζ + $i\eta$ and $\sigma = x\ iy$, then

$$\frac{\frac{d}{d\zeta}|\zeta-\sigma|}{|\zeta-\sigma|}=\frac{\frac{d}{d\zeta}(\sqrt{(\xi-x)^2+(\eta-y)^2})}{(\sqrt{\xi-x)^2+(\eta-y)^2})}$$

$$\frac{\frac{1}{2}\{\frac{\partial}{\partial\xi}(\sqrt{(\xi-x)^2+(\eta-y)^2})-i\frac{\partial}{\partial\eta}(\sqrt{(\xi-x)^2+(\eta-y)^2})\}}{(\sqrt{(\xi-x)^2+(\eta-y)^2})}$$

$$=\frac{(\xi-x)-i(\eta-y)}{2[(\xi-x)^2+(\eta-y)^2]}$$

$$=\frac{(\xi-i\eta)-(x-iy)}{2[(\xi-x)^2+(\eta-y)^2]}$$

$$=\frac{\overline{\zeta-\overline{\sigma}}}{2(\zeta-\sigma)\overline{(\zeta-\sigma)}}$$

$$=\frac{1}{2(\zeta-\sigma)}.$$

Lemma A.2.

$$\frac{1}{2(\zeta-\sigma)}=\begin{cases}\sum_{n=0}^{\infty}-\frac{\zeta^n}{2\sigma^{n+1}}, & |\zeta|<|\text{sigma}|\\ \sum_{n=0}^{\infty}\frac{\sigma^n}{2\zeta^{n+1}}, & |\zeta|>|\sigma|\end{cases} \tag{A.2}$$

Proof. Since

$$\frac{1}{2(\zeta-\sigma)}=\frac{1}{2\sigma}(\frac{1}{(\frac{\zeta}{\sigma})-1}),\sigma\neq 0$$

$$=-\frac{1}{2\sigma}(\frac{1}{1-(\frac{\zeta}{\sigma})})$$

$$=-\frac{1}{2\sigma}\sum_{n=0}^{\infty}(\frac{\zeta}{\sigma})^n,|\zeta|<|\sigma \tag{A.3}$$

$$=\sum_{n=0}^{\infty}-\frac{\zeta^n}{2\sigma^{n+1}},|\zeta|<|\sigma|$$

also

$$\frac{1}{2(\zeta-\sigma)} = \frac{1}{2\zeta}\left(\frac{1}{1-(\frac{\sigma}{\zeta})}\right)$$

$$= \frac{1}{2\zeta}\sum_{n=0}^{\infty}(\frac{\sigma}{\zeta}), \quad |\zeta|>|\sigma| \tag{A.4}$$

$$= \sum_{n=0}^{\infty}\frac{\sigma^n}{2\zeta^{(n+1)}}, \quad |\zeta|>|\sigma|$$

Lemma A.3.

$$\log\ |\zeta-\sigma| = \begin{cases} \sum_{n=0}^{\infty} -\frac{1}{2(n+1)}(\frac{\zeta}{\sigma})^{n+1}, & |\zeta|<|\sigma| \\ \frac{1}{2}\log\zeta + \sum_{n=1}^{\infty} -\frac{1}{2n}(\frac{\sigma}{\zeta})^n, & |\zeta|<|\sigma| \end{cases} \tag{A.5}$$

Proof. Since

$$\log\ |\zeta-\sigma| = \int \frac{1}{2(\zeta-\sigma)}d\zeta \qquad \text{see Lemma A.1)}$$

$$= \begin{cases} \int \sum_{n=0}^{\infty} -\frac{\zeta^n}{2\sigma^{n+1}}\,d\zeta, & |\zeta|<|\sigma|; \\ \int \sum_{n=0}^{\infty} \frac{\sigma^n}{2\zeta^{n+1}}\,d\zeta, & |\zeta|>|\sigma| \text{ (see Lemma A.2)} \end{cases} \tag{A.6}$$

$$= \begin{cases} \sum_{n=0}^{\infty} -\frac{1}{2(n+1)}(\frac{\zeta}{\sigma})^{n+1}, & |Z|<|\sigma|; \\ \frac{1}{2}\log\zeta + \sum_{n=1}^{\infty} -\frac{1}{2n}(\frac{\sigma}{\zeta})^n, & |\zeta|>|\sigma|. \end{cases}$$

Lemma A.4. If $\sigma = re^{i\alpha}$, then

$$\log\ |\zeta - re^{i\alpha}| = \begin{cases} \sum_{n=1}^{\infty} -\frac{\zeta^n}{2nr^n}e^{-in\alpha}, & |\zeta|<|\sigma| \\ \frac{1}{2}\log\zeta + \sum_{n=1}^{\infty} -\frac{r^n}{2n\zeta^n}e^{in\alpha}, & |\zeta|>|\sigma| \end{cases} \tag{A.7}$$

Proof. We have from Lemma A.3,

$$\log\,|\zeta - re^{i\alpha}| = \begin{cases} \displaystyle\sum_{n=1}^{\infty} -\frac{1}{2(n+1)}\frac{\zeta^{n+1}}{r^{n+1}e^{i(n+1)\alpha}}, & |\zeta|<|\sigma| \\ \displaystyle\frac{1}{2}\log\zeta + \sum_{n=1}^{\infty} -\frac{1}{2n}\frac{r^n e^{in\alpha}}{\zeta^n}, & |\zeta|>|\sigma| \end{cases}$$

$$= \begin{cases} \displaystyle\sum_{n=0}^{\infty} -\frac{\zeta^{n+1}}{2(n+1)r^{n+1}}e^{-i(n+1)\alpha}, & |\zeta|<|\sigma|; \\ \displaystyle\frac{1}{2}\log\zeta + \sum_{n=1}^{\infty} -\frac{r^n}{2n\zeta^n}e^{in\alpha}, & |\zeta|<|\sigma|; \end{cases} \tag{A.8}$$

$$= \begin{cases} \displaystyle\sum_{n=1}^{\infty} -\frac{\zeta^2}{2nr^n}e^{-in\alpha}, & |\zeta|<|\sigma|; \\ \displaystyle\frac{1}{2}\log\zeta + \sum_{n=1}^{\infty} -\frac{r^n}{2n\zeta^2}e^{in\alpha}, & |\zeta|<|\sigma|. \end{cases}$$

خوارزمية سريعة لحل معادلة بواسون في المستوى المركب

داود سليمان مشاط

قسم الرياضيات ، كلية العلوم ، جامعة الملك عبد العزيز

جـــــــدة – المملكة العربية السعودية

المستخلص . تم استعراض خوارزمية سريعة تجد بدقة مؤثرًا تكامليًا يظهر في حل بعض المعادلات التفاضلية الجزئية داخل دائرة الوحدة . وهذه الخوارزمية تعتمد على بعض العلاقات التكرارية في فراغ الدوال المتصلة المركبة وعلى تحويلات فورير السريعة .